**ROTATIONAL DYNAMICS**

- **Torque**
  - Torque ($\tau$) is a measure of effectiveness of a force in producing rotation of an object about an axis.
  - The Torque is measured by the product of the force and the perpendicular distance from the axis of rotation to the line along which the force acts.
  - This perpendicular distance is often referred to as the **lever arm**.

\[
\tau = Fl = Fr \sin \theta
\]

- The SI unit of torque the meters-newtons (m.N)

**ROTATIONAL DYNAMICS...**

- **Statics**
  - As the name implies, “statics” is the study of systems that don’t move such as:
    - Ladders, sign-posts, balanced beams, buildings, bridges, etc...
  - Example: What are all of the forces acting on a car parked on a hill?
  - We use Newton’s Second law then resolve into x and y components

\[
\sum \vec{F} = m\vec{a}
\]

\[
x: f_s - mg \sin \theta = 0
\]

\[
y: F_N - mg \cos \theta = 0
\]
ROTATIONAL DYNAMICS...

• Statics...
  – Now consider a plank of length ($L$) and mass ($M = 5$kg) suspended by two strings as shown. We want to find the tension in each string:

$$\sum \vec{F} = m\vec{a}$$

$$T_1 + T_2 - Mg = 0$$

– This is no longer enough to solve the problem.
– One equation with two unknown variables

We have one equation and we need more information !!

ROTATIONAL DYNAMICS...

• Statics...
  – We do have more information:
  – We know that the plank is not rotating !
  – We know that the sum of all torques is zero, and this is true about any axis we choose.

$$\sum \tau = 0$$

– Choose the rotation axis to be out of the page and through the center of mass:

$$\bar{\tau}_{T_1} + \bar{\tau}_{T_2} + \bar{\tau}_{mg} = 0$$

$$-T_1 \left( \frac{L}{2} \right) + T_2 \left( \frac{L}{4} \right) + 0 = 0 \Rightarrow \text{This is my second equation.}$$
ROTATIONAL DYNAMICS...

- Statics...
  - We already found

\[ T_1 + T_2 - Mg = 0 \]
\[-T_1 \left( \frac{L}{2} \right) + T_2 \left( \frac{L}{4} \right) + 0 = 0 \]

- Solving for \( T_1 \) and \( T_2 \) we obtain

\[
T_1 = \frac{1}{3} Mg \\
T_2 = \frac{2}{3} Mg
\]

\[
\Rightarrow \begin{cases} 
T_1 = 16.33 N \\
T_2 = 32.66 N
\end{cases}
\]

ROTATIONAL DYNAMICS...

- Rigid Objects in Equilibrium
  - A rigid body is said to be in static equilibrium if;
  - 1) The vector sum of the forces acting on it equals to zero.

\[
\sum \vec{F} = 0
\]

- 2) The vector sum of all torques acting about an axis perpendicular to the plane of the forces is zero.

\[
\sum \tau = 0
\]
**ROTATIONAL DYNAMICS...**

- Ladder on a smooth wall (modified from the first version)
  - A person (mass $M=70\text{kg}$) is climbing a ladder (length $L=6\text{m}$, mass $m=20\text{kg}$) that leans against a smooth wall (no friction between wall and ladder). A frictional force $f_s$ between the ladder and the floor keeps it from slipping. The angle between the ladder and the wall is $\theta=30^\circ$.
  - What is the magnitude of $f_s$ so that the person can reach the top of the ladder?

\[
\sum \vec{F} = 0
\]
\[
\sum \vec{\tau} = 0
\]

- Consider all of the forces acting on the ladder. In addition to force of gravity and friction, there will be normal forces $N_F$ and $N_W$ by the floor and wall on the ladder.

\[
\sum \vec{F} = 0
\]
\[
x : f_s - N_W = 0 \quad \Rightarrow \quad f_s = N_W
\]
\[
y : N_F - Mg - mg = 0 \quad \Rightarrow \quad N_F = (M + m)g
\]
ROTATIONAL DYNAMICS...

• Ladder on a smooth wall...
  - Since \( f_s = N_W \), the best choice for the axis of rotation is the bottom of the ladder: Two unknowns, \( N_f \) and \( f_s \) can be eliminated from the torque equation.

\[
\sum \tau = 0 \\
- Mg \frac{L}{2} \sin \theta - mgL \sin \theta + N_W L \sin(90 - \theta) = 0
\]

\[
\Rightarrow N_W \cos \theta = \frac{1}{2} Mg \sin \theta + mgsin \theta
\]

Solving for \( N_W \) we obtain

\[ N_W = f_s = 452.6N \]

ROTATIONAL DYNAMICS...

• Hanging Lamp
  - A lamp of mass \( M = 280 \text{ kg} \) hangs from the end of plank of mass \( m = 25.0 \text{ kg} \) and length \( L = 2.20 \text{ m} \). One end of the plank is held to a wall by a hinge, and the other end is supported by a massless string that makes an angle \( \theta = 30^\circ \) with the plank. (The hinge supplies a force to hold the end of the plank in place.)
  - a) What is the tension in the string?
  - b) What are the forces supplied by the hinge on the plank?

\[
\sum \vec{F} = 0 \\
\sum \tau = 0
\]

If we choose the rotation axis to be through the hinge then the hinge forces \( F_x \) and \( F_y \) will not enter into the torque equation:
**ROTATIONAL DYNAMICS...**

- Hanging Lamp...

\[ \sum \tau = 0 \quad \Rightarrow \quad Lmg + \frac{L}{2}mgLT \sin \theta = 0 \]

Solving for \( T \) we obtain: \( \Rightarrow T = 5733N \)

\[ \sum \vec{F} = 0 \Rightarrow \begin{cases} x : -F_x + T \cos \theta = 0 \\ y : F_y + T \sin \theta - Mg - mg = 0 \end{cases} \]

Solving for \( F_x \) and \( F_y \) we obtain:

\[ F_x = 4965N \]
\[ F_y = 123N \]

---

**ROTATIONAL DYNAMICS...**

- Problem 1

  - A 1kg ball is hung at the end of a rod 1m long. The system balances at a point on the rod 0.25m from the end holding the mass. What is the mass of the rod?

Let us choose the axis of rotation to be through the pivot point so that the unknown normal force is eliminated from the torque equation

\[ \sum \tau = 0 \]
\[ \tau_{Mg} + \tau_{mg} + \tau_{FN} = 0 \]
\[ Mg\left(\frac{L}{4}\right) - mg\left(\frac{L}{4}\right) + 0 = 0 \]
\[ m = 1kg \]
ROTATIONAL DYNAMICS...

- Torque and Rotational Inertia
  - When an unbalanced torque acts on an object, it tends to change an object’s state of rotation, i.e., it produces an angular acceleration or deceleration. However, the magnitude of the angular acceleration or deceleration depends on the object's moment of inertia (I) as well as the magnitude of the torque (τ).

\[ \sum \tau = I \alpha \]

- Moment of Inertia or Rotational Inertia
  - Just as objects tend to resist any change of translational motion, an object tends to resist any change in its rotational motion.
  - The tendency to resist any object in translational motion is referred to as inertia and is measured by measuring an object’s mass in kg. Moment of Inertia is the measure of the tendency of an object to resist any change in its state of rotation.
  - The moment of inertia is determined by calculating the sum of the moment of inertia of the particles that make up the object. It is determined not only by the mass of the object but also by the distribution of the mass about the axis of rotation.
  - The moment of inertia is determined by applying the formula

\[ I = \sum mr^2 \]

  - where I is the symbol for the moment of inertia in kg.m².
  - r is the distance from the axis of rotation to the particular particle.
**ROTATIONAL DYNAMICS...**

- Moment of Inertia of Certain Solids of Uniform Composition

<table>
<thead>
<tr>
<th>Object Description</th>
<th>Moment of Inertia</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thin-walled hollow cylinder or hoop</td>
<td>$I = MR^2$</td>
</tr>
<tr>
<td>Solid cylinder or disk</td>
<td>$I = \frac{1}{2}MR^2$</td>
</tr>
<tr>
<td>Solid sphere, axis through center</td>
<td>$I = \frac{2}{5}MR^2$</td>
</tr>
<tr>
<td>Thin rod, axis perpendicular to rod and passing through one end</td>
<td>$I = \frac{1}{3}ML^2$</td>
</tr>
<tr>
<td>Thin rod, axis perpendicular to rod and passing through center</td>
<td>$I = \frac{1}{12}ML^2$</td>
</tr>
</tbody>
</table>

**Rotational Kinetic Energy**

- A rotating object has the ability to do work and therefore has energy. The energy is in the form of rotational kinetic and is given by the formula

\[
KE = \frac{1}{2} I \omega^2
\]

**Rotational Kinetic Energy**

- The total kinetic energy of an object that has both translational as well as rotational kinetic energy, for example, a wheel on a moving car, can be expressed as follows:

\[
KE = \frac{1}{2} mv^2 + \frac{1}{2} I \omega^2
\]

- where $v$ is the linear velocity of the center of mass and $I$ is the object’s moment of inertia about an axis through an object’s center of mass.
**ROTATIONAL DYNAMICS...**

- **Problem 2**
  - One end of a string is attached to a 1.0 kg object while the other end is wrapped around a solid cylinder. The 1.0 kg object is released from rest and accelerates downward while the solid cylinder rotates about a point located at its center. The mass of the solid cylinder is 2.0 kg and its radius is 0.030 m.
  - a) Use rotational dynamics to determine the rate of acceleration of the object and the tension in the string.

\[
\sum \vec{F} = 0 \quad \Rightarrow m_1g - T = m_1a \\
\sum \tau = I\alpha \quad \Rightarrow rT \sin 90^\circ = \left(\frac{1}{2}m_2r^2\right)\alpha \\
T = m_1g - m_1a \\
\alpha = \frac{a}{r} \quad \Rightarrow \quad \begin{cases} m_1g - m_1a = \left(\frac{1}{2}m_2r^2\right)\frac{a}{r} \\
a = 4.9 \text{ m/s}^2 \end{cases}
\]

**ROTATIONAL DYNAMICS...**

- **Problem 3**
  - A small ball of mass \(m\) and radius \(R\) rolls from rest and without slipping along the loop apparatus shown in the diagram. Determine the minimum high \(h\) from which the ball can be released and still negotiate the loop. Express your answer in terms of radius of the loop \(r\). Assume that the radius of the loop is much greater than the radius of the sphere.

\[
KE_i + PE_i = KE_f + PE_f \quad \Rightarrow 0 + mgh = \frac{1}{2}mv_f^2 + \frac{1}{2}I\omega^2 + mgh_f \\
I = \frac{2}{5}mR^2 \quad \Rightarrow mgh = \frac{1}{2}mv_f^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v^2}{r^2}\right) + mg(2r) \\
\Rightarrow 0 + gh = \frac{1}{2}(rg) + \frac{1}{2}(rg) + g(2r) \\
\text{Solving for } h, \text{ we obtain} \quad \Rightarrow h = 2.7r
\]
• Angular Momentum
  - Angular momentum ($L$) is a quantity that is found from the product of an objects moment of inertia ($I$) and angular velocity ($\omega$).
  
  $$L = I\omega$$

  - The units for angular momentum is kg.m$^2$/s.

• Conservation of Angular Momentum
  - A torque tends to change an objects angular momentum and the relationship between torque and angular momentum is described by the following equation:

  $$\tau = \frac{\Delta L}{\Delta t}$$

  - The law of conservation of angular momentum states that in the absences of a net torque acting on an object, the objects angular momentum must remain constant in both magnitude and direction, i.e.

  $$\tau = \frac{\Delta L}{\Delta t}$$
ROTATIONAL DYNAMICS...

• Rotating Disks
  - A disk of mass $M$ and radius $R$ rotates around the $z$ axis with angular velocity $\omega_0$. A second identical disk, initially not rotating, is dropped on top of the first. There is friction between the disks, and eventually they rotate together with angular velocity $\omega_f$.

\[
I_i \omega_i = I_f \omega_f \quad \Rightarrow \quad \left( \frac{1}{2} MR^2 \right) \omega_0 = \left( \frac{1}{2} (2M) R^2 \right) \omega_f
\]
\[
\Rightarrow \quad \omega_f = \frac{1}{2} \omega_0
\]

• Rotating Stool
  - A student sits on a rotating stool with his arms extended and a weight in each hand. The total moment of inertia is $I_i$, and he is rotating with angular speed $\omega_i$. He then pulls his hands in toward his body so that the moment of inertia reduces to $I_f$. What is his final angular speed $\omega_f$?

\[
I_i \omega_i = I_f \omega_f
\]
Notice that when $I$ decreases $\omega$ increases and visa-versa.