1.3 Division of Polynomials; Remainder and Factor Theorems

Objectives
- Perform long division of polynomials
- Perform synthetic division of polynomials
- Apply remainder and factor theorems

In previous courses, you may have learned how to factor polynomials using various techniques. Many of these techniques apply only to special kinds of polynomial expressions. For example, the previous two sections of this chapter dealt only with polynomials that could easily be factored to find the zeros and \( x \)-intercepts.

In order to be able to find zeros and \( x \)-intercepts of polynomials which cannot readily be factored, we first introduce long division of polynomials. We will then use the long division algorithm to make a general statement about factors of a polynomial.

**Long division of polynomials**

Long division of polynomials is similar to long division of numbers. When dividing polynomials, we obtain a quotient and a remainder. Just as with numbers, if a remainder is 0, then the divisor is a factor of the dividend.

**Example 1** Determining Factors by Division

Divide to determine whether \( x - 1 \) is a factor of \( x^2 - 3x + 2 \).

**Solution** Here \( x^2 - 3x + 2 \) is the dividend and \( x - 1 \) is the divisor.

**Step 1** Set up the division as follows:

\[
\begin{array}{c|cc}
\text{divisor} & x - 1 & \text{\(n\)} & \text{\(x^2 - 3x + 2\)} & \text{dividend} \\
\hline
\end{array}
\]

**Step 2** Divide the leading term of the dividend \( x^2 \) by the leading term of the divisor \( x \). The result \( x \) is the first term of the quotient, as illustrated below.

\[
\begin{array}{c|cc}
\text{divisor} & x - 1 & \text{\(x\)} & \text{\(x^2 - 3x + 2\)} & \text{dividend} \\
\hline
\end{array}
\]

\[
\begin{array}{c|cc}
\text{first term of quotient} & \text{\(x\)} & \text{\(x\)} & \text{\(x\)} & \text{\(x\)} \\
\hline
\text{\(x - 1\)} & \text{\(\overline{x^2 \quad - 3x + 2}\)} & \text{\(x^2 \quad - 3x + 2\)} & \text{\(x^2 \quad - 3x + 2\)} & \text{\(x^2 \quad - 3x + 2\)} \\
\end{array}
\]
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Step 3 Take the term of the quotient that was just found \((x)\), and multiply it by the divisor, which gives \(x^2 - x\). Put this result in the second row.

\[
\begin{array}{c|ccc}
& x & x^2 & -3x + 2 \\
\hline
x - 1 & x^2 & -x & \leftarrow \text{multiply } x \text{ by divisor} \\
\hline
& x^2 & -x & \\
\end{array}
\]

Step 4 Subtract the second row from the first row (which gives \(-2x\)), and treat the resulting expression \((-2x + 2)\) as though it were a new dividend, just like in long division of numbers.

\[
\begin{array}{c|ccc}
& x & x^2 & -3x + 2 \\
\hline
x - 1 & x^2 & -x & \leftarrow (x^2 - 3x + 2) - (x^2 - x) \text{ (watch your signs!)} \\
\hline
& x^2 & -x & \leftarrow -2x + 2 \\
\end{array}
\]

Step 5 Continue as in Steps 1-4, but in Step 2 divide the leading term of the expression in the bottom row (i.e., the leading term of \(-2x + 2\)) by the leading term of the divisor. This result \((-2)\) is the second term of the quotient.

\[
\begin{array}{c|ccc}
& x & x^2 & -3x + 2 \\
\hline
x - 1 & x^2 & -x & \leftarrow -2x + 2 \\
\hline
& x^2 & -x & \leftarrow -2x + 2 \\
\end{array}
\]

Thus dividing \(x^2 - 3x + 2\) by \(x - 1\) gives \(x - 2\) as the quotient and 0 as the remainder, which means that the \((x - 1)\) is a factor of \((x^2 - 3x + 2)\).

Check your answer: \((x - 2)(x - 1) = x^2 - 3x + 2\).
Check It Out 1

Find the quotient and remainder when the polynomial \(x^2 + x - 6\) is divided by \(x - 2\).

We can make the following statement about the relationship between the dividend, divisor, quotient and remainder:

\[
\text{(divisor \times quotient) + remainder = dividend.}
\]

This is a very important result and is stated formally as follows.

**The Division Algorithm**

Let \(p(x)\) be a polynomial divided by a nonzero polynomial \(d(x)\). Then, there are a quotient polynomial \(q(x)\) and a remainder polynomial \(r(x)\) such that

\[
p(x) = d(x)q(x) + r(x), \quad \text{or equivalently, } \frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)},
\]

where either \(r(x) = 0\) or the degree of \(r(x)\) is less than the degree of \(d(x)\).

The following illustrates the relationship between factors and remainders.

**Factors and remainders**

Let a polynomial \(p(x)\) be divided by a nonzero polynomial \(d(x)\), with a quotient polynomial \(q(x)\) and a remainder polynomial \(r(x)\). If \(r(x) = 0\), then \(d(x)\) and \(q(x)\) are both factors of \(p(x)\).

The next example will show how polynomial division can be used to factor a polynomial that cannot be factored by straightforward methods that you may already know.

**Example 2  Long Division of Polynomials**

Find the quotient and remainder when \(2x^4 + 7x^3 + 4x^2 - 7x - 6\) is divided by \(2x + 3\).
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Solution  We follow the same steps as before, but shall condense them in this example.

Step 1  
- Divide the leading term of the dividend \((2x^4)\) by the leading term of the divisor \((2x)\). The result, \(x^3\), is the first term of the quotient.
- Multiply the term of the quotient that was just found, \(x^3\), by the divisor, and put the result, \(2x^4 + 3x^3\), in the second row.
- Subtract the second row from the first row, just like in the division of numbers.

\[
\begin{array}{r}
2x + 3 \underline{| x^3 \, + \, 2x^2 \, + \, 7x^3 \, + \, 4x^2 \, - \, 7x \, - \, 6} \\
\hline
2x^4 + 3x^3 \\
\hline
4x^5 + 4x^2 - 7x - 6 & \quad \text{← multiply } x^3 \text{ by divisor}
\end{array}
\]

\[
\begin{array}{r}
2x + 3 \underline{| x^3 \, + \, 2x^2 \, + \, 7x^3 \, + \, 4x^2 \, - \, 7x \, - \, 6} \\
\hline
2x^4 + 3x^3 \\
\hline
4x^5 + 4x^2 - 7x - 6 & \quad \text{← subtraction}
\end{array}
\]

Step 2  Divide the leading term of the expression in the bottom row, \(4x^3\), by the leading term of the divisor. Multiply the result, \(2x^2\), by the divisor, and subtract:

\[
\begin{array}{r}
2x + 3 \underline{| x^3 \, + \, 2x^2 \, + \, 7x^3 \, + \, 4x^2 \, - \, 7x \, - \, 6} \\
\hline
2x^4 + 3x^3 \\
\hline
4x^5 + 4x^2 - 7x - 6 & \quad \text{← multiply } 2x^2 \text{ by divisor}
\end{array}
\]

\[
\begin{array}{r}
2x + 3 \underline{| x^3 \, + \, 2x^2 \, + \, 7x^3 \, + \, 4x^2 \, - \, 7x \, - \, 6} \\
\hline
2x^4 + 3x^3 \\
\hline
4x^5 + 4x^2 - 7x - 6 & \quad \text{← subtraction}
\end{array}
\]

Step 3  Divide the leading term of the expression in the bottom row, \(-2x^2\), by the leading term of the divisor. Multiply the result, \(-x\), by the divisor, and subtract:

\[
\begin{array}{r}
2x + 3 \underline{| x^3 \, + \, 2x^2 \, + \, 7x^3 \, + \, 4x^2 \, - \, 7x \, - \, 6} \\
\hline
2x^4 + 3x^3 \\
\hline
4x^5 + 4x^2 - 7x - 6 & \quad \text{← multiply } -x \text{ by divisor}
\end{array}
\]

\[
\begin{array}{r}
2x + 3 \underline{| x^3 \, + \, 2x^2 \, + \, 7x^3 \, + \, 4x^2 \, - \, 7x \, - \, 6} \\
\hline
2x^4 + 3x^3 \\
\hline
4x^5 + 4x^2 - 7x - 6 & \quad \text{← subtraction (be careful with signs!)}
\end{array}
\]
**Step 4** Divide the leading term of the expression in the bottom row, \(-4x\), by the leading term of the divisor. Multiply the result, \(-2\), by the divisor, and subtract:

\[
\begin{array}{c|ccccc}
& x^3 + 2x^2 - x & -2 \\
\hline
2x + 3 & 2x^4 + 7x^3 + 4x^2 - 7x & -6 \\
\hline
& 2x^4 + 3x^3 & \\
\hline
& 4x^3 + 4x^2 - 7x & -6 \\
\hline
& 4x^3 + 6x^2 & \\
\hline
& -2x^2 - 7x & -6 \\
\hline
& -2x^2 - 3x & \\
\hline
& -4x & -6 & \text{multiply } -2 \text{ by divisor} \\
\hline
& 0 & \text{subtraction (be careful with signs!)}
\end{array}
\]

Thus dividing \(2x^4 + 7x^3 + 4x^2 - 7x - 6\) by \(2x + 3\) gives a quotient, \(q(x)\), of \(x^3 + 2x^2 - x - 2\) with a remainder \(r(x)\), of 0, or equivalently,

\[
\frac{2x^4 + 7x^3 + 4x^2 - 7x - 6}{2x + 3} = x^3 + 2x^2 - x - 2.
\]

You can check that \((2x + 3)(x^3 + 2x^2 - x - 2) = 2x^4 + 7x^3 + 4x^2 - 7x - 6\). □

**Check It Out 2**

Find the quotient and remainder when \(6x^3 - x^2 - 3x + 1\) is divided by \(2x - 1\). □

Thus far, all the long division problems illustrated have not produced a remainder. The next example will give an illustration of long division with a remainder.

**Example 3  Long Division with Remainder**

Find the quotient and remainder when \(p(x) = 6x^3 + x - 1\) is divided by \(d(x) = x + 2\). Write your answer in the form \(\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}\).
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Solution The steps for long division should now be pretty clear to you.

- Note that the expression $6x^3 + x - 1$ does not have an $x^2$ term; i.e., its coefficient is 0. Thus, when we set up the division, we write the $x^2$ term as $0x^2$. We then divide $6x^3$ by the leading term of the divisor:

$$
\begin{array}{c|ccc}
& 6x^2 & 0x^2 & + x - 1 \\
\hline
x + 2 & 6x^3 & + 12x^2 & \\
\hline
 & 6x^3 & + 12x^2 & \\
6x^3 + 12x^2 & -12x^2 & + x - 1 & \leftarrow \text{multiply } 6x^2 \text{ by divisor} \\
\hline
 & -12x^2 & -24x & \leftarrow \text{subtract} \\
\hline
 & 25x - 1 & & \leftarrow \text{multiply } -12x \text{ by divisor} \\
25x & -1 & & \leftarrow \text{subtract} \\
\hline
\end{array}
$$

- Next, we divide $-12x^2$ by the leading term of the divisor:

$$
\begin{array}{c|ccc}
& 6x^2 & -12x & 25 \\
\hline
x + 2 & 6x^3 + 0x^2 & + x & -1 \\
\hline
 & 6x^3 & + 12x^2 & \\
6x^3 + 12x^2 & -12x^2 & + x & -1 \\
\hline
 & -12x^2 & -24x & \leftarrow \text{multiply } -12x \text{ by divisor} \\
\hline
 & 25x & +50 & \leftarrow \text{multiply } 25 \text{ by divisor} \\
25x & -51 & & \leftarrow \text{subtract} \\
\hline
\end{array}
$$

- Finally, we divide $25x$ by the leading term of the divisor:

$$
\begin{array}{c|ccc}
& 6x^2 & -12x & +25 \\
\hline
x + 2 & 6x^3 + 0x^2 & + x & -1 \\
\hline
 & 6x^3 & + 12x^2 & \\
6x^3 + 12x^2 & -12x^2 & + x & -1 \\
\hline
 & -12x^2 & -24x & \leftarrow \text{multiply } -12x \text{ by divisor} \\
\hline
 & 25x & -51 & \leftarrow \text{subtract} \\
\hline
\end{array}
$$

Thus, the quotient $q(x) = 6x^2 - 12x + 25$ and the remainder, $r(x)$, is $-51$, or equivalently,

$$
\frac{6x^3 + x - 1}{x + 2} = 6x^2 - 12x + 25 - \frac{51}{x + 2}.
$$

In this example, long division yields a nonzero remainder. Thus $x + 2$ is not a factor of $6x^3 + x - 1$. 
\qed
Check It Out 3

Find the quotient and remainder when $3x^3 + x^2 - 1$ is divided by $x + 1$. Write your answer in the form $\frac{p(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$. □

Synthetic division

Synthetic division is a compact way of dividing polynomials when the divisor is of the form $x - c$. Instead of writing out all the terms of the polynomial, we work only with the coefficients. We illustrate this shorthand form of polynomial division with the problem from Example 3.

**Example 4**  Synthetic Division

Use synthetic division to divide $6x^3 + x - 1$ by $x + 2$.

**Solution**  Since the divisor is of the form $x - c$, we can use synthetic division. Note that $c = -2$.

**Step 1**  Write down the coefficients of the dividend in a row, from left to right, and then place the value of $c$ (which is $-2$) in that same row, to the left of the leading coefficient of the dividend:

\[
\begin{array}{cccc}
\text{value of } c = -2 & \rightarrow & -2 & 6 & 0 & 1 & -1 & \leftarrow \text{list coefficients of the dividend}
\end{array}
\]

**Step 2**  Bring down the leading coefficient of the dividend, 6, and then multiply it by $c$, which is $-2$:

\[
\begin{array}{cccc}
-2 & | & 6 & 0 & 1 & -1
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & | & \left[-12\right] & \leftarrow & \text{b) then multiply 6 by } -2
\end{array}
\]

\[
\begin{array}{cccc}
a) \text{first bring down 6 } & \rightarrow & 6 & \leftarrow
\end{array}
\]

**Step 3**  Place the result, $-12$, below the coefficient of the next term of the dividend, $0$, and add:

\[
\begin{array}{cccc}
-2 & | & 6 & 0 & 1 & -1
\end{array}
\]

\[
\begin{array}{cccc}
\downarrow & | & \left[-12\right] & \leftarrow & \text{add 0 and } -12
\end{array}
\]
Step 4  Apply Steps 2(b) and 3 to the result, which is −12:

\[
\begin{array}{cccc}
-2 & | & 6 & 0 & 1 & -1 \\
\hline
 & & -12 & 24 & & \\
6 & & -12 & 25 & & \\
- & & & 25 & & \leftarrow \text{ add } 1 \text{ and } 24
\end{array}
\]

Step 5  Apply Steps 2b and 3 to the result, which is 25:

\[
\begin{array}{cccc}
-2 & | & 6 & 0 & 1 & -1 \\
\hline
 & & -12 & 24 & & \\
6 & & -12 & 25 & & \\
- & & & -51 & & \leftarrow \text{ add } -1 \text{ and } -50
\end{array}
\]

Step 6  Read off the coefficients of the quotient from the bottom row:

\[q(x) = 6x^2 - 12x + 25\]

The remainder, \(r(x)\), is −51, the last number in the bottom row. The result is the same as the one found by long division, but the procedure is much more compact.

Check It Out 4

Use synthetic division to divide \(-2x^3 + 3x^2 - 1\) by \(x - 1\).

The remainder and factor theorems

We now examine an important connection between a polynomial \(p(x)\) and the remainder we get when \(p(x)\) is divided by \(x - c\). In Example 3, division of \(p(x) = 6x^3 + x - 1\) by \(x + 2\) yielded a remainder of −51. Also, \(p(-2) = -51\). Is this just a coincidence? Not really – it’s actually one of the many interesting properties of polynomial division. The remainder theorem is formally stated as follows.

The Remainder Theorem

When a polynomial \(p(x)\) is divided by \(x - c\), the remainder is equal to the value of \(p(c)\).

Since the remainder is equal to \(p(c)\), synthetic division provides a quick way to evaluate \(p(c)\). This is illustrated in the next example.
Example 5 Applying the Remainder Theorem

Let \( p(x) = -2x^4 + 6x^3 + 3x - 1 \). Use synthetic division to evaluate \( p(2) \).

Solution From the remainder theorem, \( p(2) \) is the remainder obtained when \( p(x) \) is divided by \( x - 2 \). Following the steps outlined in Example 4, we have the following.

\[
\begin{array}{c|cccc}
2 & -2 & 6 & 0 & 3 & -1 \\
 & & -4 & 4 & 8 & 22 \\
\hline
 & -2 & 2 & 4 & 11 & 21 \\
\end{array}
\]

Since the remainder is 21, we know that \( p(2) = 21 \).

Check It Out 5

Let \( p(x) = 3x^4 - x^2 + 3x - 1 \). Use synthetic division to evaluate \( p(-2) \).

We also have the following factor theorem, which is a direct result of the remainder theorem.

The Factor Theorem

The term \( x - c \) is a factor of a polynomial \( p(x) \) if and only if \( p(c) = 0 \).

The Factor Theorem makes an important connection between zeros and factors. It states that if we have a linear factor, i.e. a factor of the form \( x - c \), of a polynomial \( p(x) \), then \( p(c) = 0 \). That is, \( c \) is a zero of the polynomial \( p(x) \). It also works the other way around: if \( c \) is a zero of the polynomial \( p(x) \), then \( x - c \) is a factor of \( p(x) \).

Example 6 Applying the Factor Theorem

Determine whether \( x + 3 \) is a factor of \( 2x^3 + 3x - 2 \).

Solution Let \( p(x) = 2x^3 + 3x - 2 \). Since \( x + 3 \) is in the form \( x - c \) we can apply the factor theorem with \( c = -3 \).

Evaluate \( p(-3) \), which gives \( p(-3) = -65 \). By the factor theorem, \( x + 3 \) is not a factor since \( p(-3) \neq 0 \).
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Check It Out 6

Determine whether \( x - 1 \) is a factor of \( 2x^3 - 4x + 2 \).

Key Points

- Let \( p(x) \) be a polynomial divided by a nonzero polynomial \( d(x) \). Then, there are a quotient polynomial \( q(x) \) and a remainder polynomial \( r(x) \) such that
  \[
p(x) = d(x)q(x) + r(x).
  \]

- The **Remainder Theorem** states that when a polynomial \( p(x) \) is divided by \( x - c \), the remainder is equal to the value of \( p(c) \).

- The **Factor Theorem** states that the term \( x - c \) is a factor of a polynomial \( p(x) \) if and only if \( p(c) = 0 \).

- Synthetic Division is a shorter way to divide polynomials when the divisor is of the form \( x - c \).

Exercises

**SKILLS** This set of exercises will reinforce the skills illustrated in this section.

Find the quotient and remainder when the first polynomial is divided by the second polynomial. You may use synthetic division wherever applicable.

1. \( 2x^2 + 13x + 15; x + 5 \) \( \text{IAE} \) \( q(x) = 2x + 3; r(x) = 0 \)
2. \( 2x^2 - 7x + 3; x - 3 \)
3. \( 2x^3 - x^2 - 8x + 4; 2x - 1 \)
4. \( 3x^3 + 2x^2 - 3x - 2; 3x + 2 \)
5. \( x^3 - 3x^2 + 2x - 4; x + 2 \)
6. \( x^3 + 2x^2 - x - 3; x - 3 \)
7. \( -3x^4 + x^2 - 2; 3x - 1 \)
8. \( 2x^4 - x^3 + x^2 - x; 2x + 1 \)
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9. $x^6 + 1; x + 1$
10. $-x^3 + x; x - 5$
11. $x^3 + 2x^2 - 5; x^2 - 2$
12. $-x^3 - 3x^2 + 6; x^2 + 1$
13. $x^5 - x^4 + 2x^3 + x^2 - x + 1; x^3 + x - 1$
14. $-2x^5 + x^4 - x^3 + 2x^2 - 1; x^3 + x^2 + 1$

Write each polynomial as $p(x) = d(x)q(x) + r(x)$, where $p(x)$ is the first polynomial and $d(x)$ is the second polynomial. You may use synthetic division wherever applicable.

15. $x^2 + x + 1; x + 1$
16. $x^2 + x + 1; x - 1$
17. $3x^3 + 2x - 8; x - 4$
18. $4x^3 - x + 4; x - 2$
19. $x^6 - 3x^5 + x^4 - 2x^2 - 5x + 6; x^2 + 2$
20. $-x^6 + 4x^5 - x^3 + x^2 + 8; x^2 + 4$

Use synthetic division to find the function values.

21. $f(x) = x^3 - 7x + 5; \text{ find } f(3) \text{ and } f(5)$
22. $f(x) = -2x^3 + 4x^2 - 7; \text{ find } f(4) \text{ and } f(-3)$
23. $f(x) = -2x^4 - 10x^3 - 3x + 10; \text{ find } f(-1) \text{ and } f(2)$
24. $f(x) = -x^4 + 3x^3 - 2x - 4; \text{ find } f(-2) \text{ and } f(3)$
25. $f(x) = x^5 - 2x^3 + 12; \text{ find } f(3) \text{ and } f(-2)$
26. $f(x) = -2x^5 + x^4 + x^2 - 2; \text{ find } f(-3) \text{ and } f(4)$
27. $f(x) = x^4 - 2x^2 + 1; \text{ find } f\left(\frac{1}{2}\right)$
28. $f(x) = -x^4 + 3x^2 - 2x; \text{ find } f\left(\frac{3}{2}\right)$
Determine whether \( d(x) \) is a factor of \( p(x) \) and justify your answer.

29. \( x^3 - 7x + 6; x - 3 \)
30. \( x^3 - 5x^2 + 8x - 4; x + 2 \)
31. \( x^3 - 7x + 6; x + 3 \)
32. \( x^3 - 5x^2 + 8x - 4; x - 2 \)
33. \( x^5 - 3x^3 + 2x - 8; x - 4 \)
34. \( -2x^4 - 7x^3 + 5; x + 2 \)
35. \( x^4 - 50; x - 5 \)
36. \( 2x^5 - 1; x - 2 \)
37. \( 3x^3 - 48x - 4x^2 + 64; x + 4 \)
38. \( x^3 + 9x + x^2 + 9; x + 1 \)

CONCEPTS This set of exercises will draw on the ideas presented in this section and your general math background.

39. Given the graph below of a polynomial function \( p(x) \), find a linear factor of \( p(x) \).

40. Consider the following graph of a polynomial function \( p(x) \).

\[ \text{(a) Evaluate } p(2). \]
(b) Is \(x - 2\) a factor of \(p(x)\)? Explain.
(c) Find the remainder when \(p(x)\) is divided by \(x - 2\).

41. Find the remainder when \(x^7 + 7\) is divided by \(x - 1\).

42. Find the remainder when \(x^8 - 3\) is divided by \(x + 1\).

43. Let \(x - \frac{1}{2}\) be a factor of a polynomial function \(p(x)\). Find \(p(1/2)\).

44. For what value(s) of \(k\) do you get a remainder of 15 when you divide \(kx^3 + 2x^2 - 10x + 3\) by \(x + 2\)?

45. For what value(s) of \(k\) do you get a remainder of \(-2\) when you divide \(x^3 - x^2 + kx + 3\) by \(x + 1\)?

46. Why is the Factor Theorem a direct result of the Remainder Theorem?